CCS Research Report 591

THE CHANCE-CONSTRAINED CRITICAL PATH FOR A LARGE CLASS OF DISTRIBUTIONS

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ABSTRACT

M. Kress proved for a special class of Location-Scale probability distributions there always exists a probability level for which the Chance Constrained Critical Path (CCCP) remains unchanged for all probabilities greater than or equal to that value. His chance constrained problem has zero-order decision rules and individual chance constraints.

This paper extends his results to most of the common probability distributions.

KEY WORDS

Chance Constrained Programming, (KR)

Chance Constrained Critical Paths

Location-Scale Distributions

Moshe Kress [1] studied the Chance Constrained Critical Path (CCCP) problem with zero order decision rule and individual chance constraints. He proved that for a class of Location-Scale probability distributions there always exists a probability level for which the CCCP remains unchanged for all probability values greater than or equal to that level. The purpose of this paper is to extend the results of [1] to most of the common probability distributions.

The CCCP problem with zero order decision rule and equal minimal probability level can be formulated as

$$P(\beta) \qquad \text{Min } u_m - u_0$$

$$\text{s.t.} \begin{cases} P_r\left(\sum_{i=0}^{m} \epsilon_{ij} u_i \geq t_j \right) \geq \beta \\ j = 1, 2, \ldots, n \end{cases}$$

where u_i's are the start times of the activities. t_j, $j = 1, 2, \ldots, n$, are the durations of activities j's which are the random variables with marginal distribution function F_j; ϵ_{ij} is the entry in row i, column j of the node-arc incidence matrix of the activity network.

$$\text{Let} \quad F_j^{-1}(\beta) = \inf \left\{ t : F_j(t) \geq \beta \right\}$$

Notice that since the distribution function is right continuous, "inf" can be changed into "min". Now we can write $P(\beta)$ into an equivalent deterministic form.

$$\text{Min } u_m - u_0$$

$$P_1(\beta) \qquad \text{s.t.:} \begin{cases} \sum_{i=0}^{m} \epsilon_{ij} u_i \geq F_j^{-1}(\beta) \\ j = 1, 2, \ldots, n \end{cases}$$

The dual problem to $P_1(\beta)$ is

$$\text{Max } \sum_{j=1}^{n} F_j^{-1}(\beta)\, x_j$$

$$D_1(\beta) \qquad \text{s.t.:} \begin{cases} \sum_{j=1}^{n} \epsilon_{ij} x_j = a_i, \quad i = 0, 1, \ldots, m \\ x_j \geq 0, \quad j = 1, 2, \ldots, n \end{cases}$$

where $a_0 = -1$, $a_m = 1$, $a_i = 0$, for $i = 1, 2, \ldots, m-1$.

Since $D_1(\beta)$ is a pure network problem, for any given $\beta \in (0, 1)$, a basic optimal solution $x^*(\beta) = \{x_1^*(\beta), x_2^*(\beta), \ldots, x_n^*(\beta)\}$ of $D_1(\beta)$ has the property $x_j^*(\beta)$ is either 0 or 1, and $\{j : x_j^*(\beta) = 1\}$ forms a critical path for the network of $D_1(\beta)$.

Let $P = \{p_k : k = 1, 2, \ldots, K\}$ denote the set of all paths from the source to the sink in the network, and $J_k = \{j : \text{arc } j \text{ is in path } P_k\}$. Then the problem $P(\beta)$ is equivalent to the following problem.

$$\overline{P}(\beta) \qquad \text{Max} \sum_{k=1}^{K} \left(\sum_{j \in J_k} F_j^{-1}(\beta) \right) y_k$$

$$\text{s.t.} \begin{cases} \sum_{k=1}^{K} y_k = 1 \\ y_k \geq 0 \\ k = 1, 2, \ldots, K \end{cases}$$

Clearly, if k^* such that

$$\sum_{j \in J_{k^*}} F_j^{-1}(\beta) = \underset{k \in I}{\text{Max}} \sum_{j \in J_k} F_j^{-1}(\beta)$$

where $I = \{1, 2, \ldots, K\}$,

then $y_{k^*} = 1$, $y_i = 0$ for $i \neq k^*$ is an optimal solution of $\overline{P}(\beta)$, and if we set

$$x_j^*(\beta) = \begin{cases} 1 & \text{if } j \in J_{k^*} \\ 0 & \text{otherwise} \end{cases}$$

$x^*(\beta) = \{ x_1^*(\beta), x_2^*(\beta), x_3^*(\beta), \ldots, x_n^*(\beta) \}$ is an optimal solution of problem $D_1(\beta)$.

Let $\quad V_k(\beta) = \sum_{j \in J_k} F_j^{-1}(\beta)$,

$$U_{k,1}(\beta) = \sum_{j \in J_k \setminus J_k \cap J_1} F_j^{-1}(\beta) ,$$

$$H(\beta) = \left\{ k : V_k(\beta) = \underset{i \in I}{\text{Max}} V_i(\beta) \right\} .$$

The following theorem gives a sufficient conditon under which the CCCP remains unchanged for all probabilities greater than or equal to a level β_0.

For the distribution function F_j , denote the support set of F_j (supp (F_j)) as the interval $[\underline{\gamma}_j, \overline{\gamma}_j]$

where $\underline{\gamma}_j = \sup \{ \gamma : F_j(\gamma) = 0 \}$

$\overline{\gamma}_j = \inf \{ \gamma : F_j(\gamma) = 1 \}$

<u>Theorem</u>: Assume marginal distribution functions F_j, $j = 1, 2, \ldots, n$, are continuous and the density functions $f_j > 0$, a.e., in supp (F_j).

Case I: $\bar{\gamma}_j < \infty$, for $j = 1, 2, \ldots, n$.

If $k^* \in H(1)$, and there exists a $0 \le \hat{\beta} < 1$ such that

$$U'_{k^*,k}(\beta) \le U'_{k,k^*}(\beta)$$

for any $\beta \in [\hat{\beta}, 1]$ and $k \in H(1)$,

then there exists a $\beta_o < 1$ such that for any $\beta \in [\beta_o, 1]$

$$V_{k^*}(\beta) \le V_k(\beta) \qquad \text{for } k \in I$$

Case 2 : $\bar{\gamma}_j = +\infty$, for $j = 1, 2, \ldots, n$.

(a) If there exists a $0 \le \hat{\beta} < 1$ such that $k^* \in H(\hat{\beta})$ and $U'_{k^*,k}(\beta) \le U'_{k,k^*}(\beta)$ for any $\beta \in [\hat{\beta}, 1]$ and $k \in I$.

or,

(b) if k^* is such that

$$\lim_{\beta \to 1^-} \frac{U'_{k^*,k}(\beta)}{U'_{k,k^*}(\beta)} > 1$$

for $k \in I$

Then, there exists a $\beta_o < 1$ such that for any $\beta \in [\beta_o, 1]$

$$V_{k^*}(\beta) \ge V_k(\beta) \qquad \text{for } k \in I$$

Proof: Case 1.

Since F_j is a continuous monotone distribution function, $f_j = F_j'$ is a Lebesgue measurable function. Since $f_j > 0$ a.e., $1/f_j = (F_j^{-1})'$ is a Lebesgue measurable function. By the condition of theorem defining Case 1, $U'_{k,k^*}(\beta) \ge U'_{k^*,k}(\beta)$ for any $\beta \in [\hat{\beta}, 1]$ and $k \in H(1)$. Therefore, for any $\beta \in [\hat{\beta}, 1]$ and $k \in H(1)$

$$0 \le \int_{\beta}^{1} \left(U'_{k,k^*}(t) - U'_{k^*,k}(t) \right) dt$$

$$= \int_{\beta}^{1} \left(V'_k(t) - V'_{k^*}(t) \right) dt$$

$$= V_k(1) - V_{k^*}(1) - V_k(\beta) + V_{k^*}(\beta)$$

By hypothesis $V_k(1) = V_{k^*}(1)$, hence

$$V_{k^*}(\beta) \geq V_k(\beta) \qquad \text{for any} \quad k \in H(1)$$

If $k \notin H(1)$, then $\qquad V_{k^*}(1) > V_k(1)$

and since $f_j > 0$ a.e. in supp (F_j), F_j^{-1} is a strictly increasing continuous function and so is the sum of these which is $V_k(\beta)$. Therefore there exists a $\beta_k < 1$ such that for any $\beta \in [\beta_k, 1]$

$$V_{k^*}(\beta) \geq V_k(\beta) .$$

Let $\beta_o = \max\left\{\hat{\beta}, \beta_k \text{ for } k \in I \setminus H(1)\right\} < 1$

Then for $\beta \in [\beta_o, 1]$, $k \in I$

$$V_{k^*}(\beta) \geq V_k(\beta)$$

Case 2.a.

For any $\beta \in [\hat{\beta}, 1]$ and $k \in I$

$$0 \leq \int_{\hat{\beta}}^{\beta} \left(U'_{k^*,k}(t) - U'_{k,k^*}(t) \right) dt$$

$$= \int_{\hat{\beta}}^{\beta} \left(V'_{k^*}(t) - V'_k(t) \right) dt$$

$$= V_{k^*}(\beta) - V_k(\beta) - V_{k^*}(\hat{\beta}) + V_k(\hat{\beta})$$

By hypothesis $k^* \in H(\hat{\beta})$, so we have

$$V_{k^*}(\beta) \geq V_k(\beta)$$

Therefore we can choose β_o at least equal to $\hat{\beta}$.

Case 2.b.

$$\text{Since } \lim_{\beta \to 1^-} \frac{U'_{k^*k}(\beta)}{U'_{k,k^*}(\beta)} > 1$$

there exists a $\hat{\beta}$ such that for any $\beta \in [\hat{\beta}, 1]$

$$\frac{U'_{k^*,k}(\beta)}{U'_{k,k^*}(\beta)} \geq 1 + \epsilon$$

that is

$$U'_{k^*,k}(\beta) - U'_{k,k^*}(\beta) \geq \epsilon\, U'_{k,k^*}(\beta)$$

Hence

$$\epsilon \int_{\hat{\beta}}^{\beta} U'_{k,k^*}(t)\,dt \leq \int_{\hat{\beta}}^{\beta} \left(U'_{k^*,k}(t) - U'_{k,k^*}(t)\right) dt = \int_{\hat{\beta}}^{\beta} \left(V'_{k^*}(t) - V'_k(t)\right) dt$$

$$\leq V_{k^*}(\beta) - V_k(\beta) - \left(V_{k^*}(\hat{\beta}) - V_k(\hat{\beta})\right)$$

Notice that if $\bar{\gamma}_j = \infty$ and $f_j > 0$ a.e. in supp (f_j) there exists a $\beta_o < 1$ such that

$$\int_{\hat{\beta}}^{\beta_o} U'_{k,k^*}(t)\,dt \geq \left(V_k(\hat{\beta}) - V_{k^*}(\hat{\beta})\right) / \epsilon$$

So, for any $\beta > \beta_o$,

$$V_{k^*}(\beta) - V_k(\beta) \geq 0$$

Hence we have obtained a β_o which satisfies our requirement.

Remark 1: From the argument of our proof, it is clear that the theorem is also true when the F_j's satisfy the assumed condition only in a interval of $[\alpha, \bar{\gamma}]$, where α is a certain real value less than $\bar{\gamma}$.

Remark 2: When the set of random variables t_i, $i = 1, 2, \ldots, n$ contains both finite support and infinite support distribution functions, then we can treat it as we did in the infinite support case.

We are now going to show that the location-scale distribution family which Kress [1] studied satisfies the hypotheses of our theorem.

The F_j, $j = 1, 2, \ldots, n$, for the Kress family are location-scale distributions with the same generating distribution $\varnothing$, that is,

$$F_j(x) = \varnothing\left(\frac{x - \mu_j}{\sigma_j}\right)$$ for $j = 1, 2, \ldots n$, where μ_j, σ_j are the location and scale parameters of F_j respectively.

The inverse of a location-scale distribution of a non-negative random variable is

$$F_j^{-1}(\beta) = \sigma_j \varnothing^{-1}(\beta) + \mu_j$$

where $\varnothing^{-1}(\beta) \geq 0$ is the βth fractile of the generating distribution $\varnothing$.

Let $S_k = \sum_{j \in J_k} \sigma_j \geq 0$, $M_k = \sum_{j \in J_k} \mu_k \geq 0$.

Then $V_k(\beta) = S_k \varnothing^{-1}(\beta) + M_k$

and the following corollary is obtained.

Corollary: Suppose F_j, $j = 1, 2, \ldots, n$, are such location-scale distributions, $\varnothing$ a non-negative continuous distribution.

Case 1. Let $\bar{\gamma} = \inf\{\gamma : \varnothing(\gamma) = 1\} < \infty$

If $K^* \in I$ such that $k^* \in H(1)$ and

$$M_{k^*} = \max_{j \in H(1)} M_j$$

Then, for any $\beta \in [0, 1]$, $k \in H(1)$

$$U'_{k^*,k}(\beta) \geq U'_{k,k^*}(\beta)$$

Case 2. Let $\bar{\gamma} = +\infty$

If $k^* \in I$ such that

$S_{k^*} \geq S_k$, for all $k \in I$ and $M_{k^*} \geq M_k$, for all $k \in H$, where $H = \left\{i : S_i = \max_{j \in I} S_j\right\}$,

then there exists a $\hat{\beta}$ such that for $\beta \in \left[\hat{\beta}, 1\right]$, $k^* \in H(\hat{\beta})$ and $U'_{k^*,k}(\beta) \geq U'_{k,k^*}(\beta)$ for $k \in I$.

Proof: Case 1. Let $k \in H(1)$, then since

$$M_{k^*} \geq M_k \text{ and } V_{k^*}(1) = S_{k^*} \varnothing^{-1}(1) + M_{k^*} = S_k \varnothing^{-1}(1) + M_k = V_k(1),$$

we have $S_{k^*} \leq S_k$.

Therefore $V'_{k^*}(\beta) = S_{k^*}\left(\varnothing^{-1}(\beta)\right)' \le S_k\left(\varnothing^{-1}(\beta)\right)' = V'_k(\beta)$

That is $U'_{k^*,k}(\beta) \le U'_{k,k^*}(\beta)$ for $k \in H(1)$

<u>Case 2</u>: By hypothesis, we know if $k \in H$

$$V_{k^*}(\beta) = S_{k^*}\varnothing^{-1}(\beta) + M_{k^*} \ge S_k\varnothing^{-1}(\beta) + M_k = V_k(\beta)$$

that is, $U_{k^*,k}(\beta) \ge U_{k,k^*}(\beta)$ (•)

If $k \bar{\in} H$, then $S_{k^*} > S_k$. Let $\mu = S_{k^*} - S_k > 0$.

Since $\bar{\gamma} = \infty$, there is a $\hat{\beta}$ such that for any $\beta \in [\hat{\beta}, 1]$, $\varnothing^{-1}(\beta) \ge \dfrac{M_k - M_{k^*}}{\mu}$

Hence, $V_{k^*}(\beta) = S_{k^*}\varnothing^{-1}(\beta) + M_{k^*}$

$$= S_k\varnothing^{-1}(\beta) + \mu\varnothing^{-1}(\beta) + M_{k^*} \qquad (\bullet\bullet)$$

$$\ge S_k\varnothing^{-1}(\beta) + M_k = V_k(\beta)$$

Combining (•) and (••), for $\beta \in [\hat{\beta}, 1]$

$$U_{k^*,k}(\beta) \ge U_{k,k^*}(\beta)$$

On the other hand, because $S_{k^*} \ge S_k$ for $k \in I$, we can directly conclude

$$V'_{k^*}(\beta) = S_{k^*}\left(\varnothing^{-1}(\beta)\right)' \ge S_k\left(\varnothing^{-1}(\beta)\right)' = V'_k(\beta)$$

which is equivalent to

$$U'_{k^*,k}(\beta) \ge U'_{k,k^*}(\beta)$$

Q.E.D.

Example 1:

Consider the project network depicted in Figure 1.

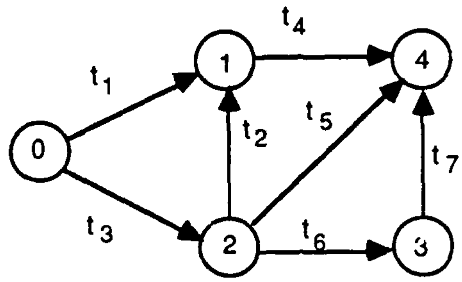

Figure 1

Let $\quad t_1 = t_2$ have distribution $U(0, 1)$

$$t_3 \sim \text{density function} \quad f_3(x) = \begin{cases} \frac{1}{2}(2-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$t_4 = t_5 \sim \text{density function} \quad f_4(x) = f_5(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$t_6 = t_7 \sim \text{density function} \quad f_6(x) = f_7(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\bar{\gamma}_i \le 2 < \infty$

By solving the pure network problem D_1 (1), we get the optimal critical paths $J_1 = \{0, 2, 1, 4\}$, $J_2 = \{0, 2, 3, 4,\}$. See Figure 2.

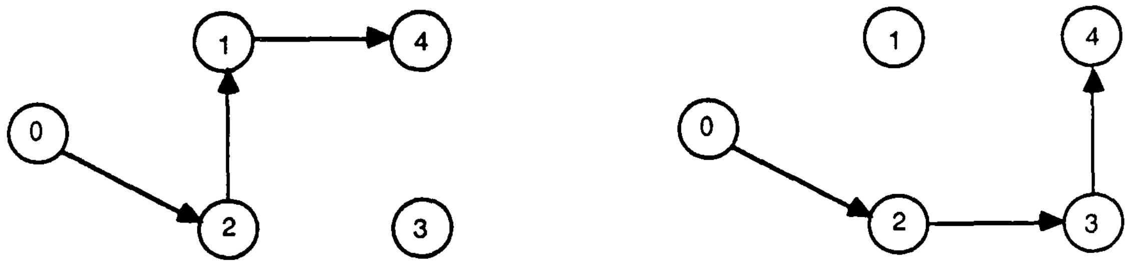

Figure 2

Since $\quad U'_{1,2}(1) = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$

and $\quad U'_{2,1}(1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

the critical path J_2 is what we are seeking. By use of the optimal solution corresponding to J_2 in D_1 (β) and computing its reduced cost, we can get the minimal probability level $\beta_o = 0$, for which CCCP remains unchanged for all probability values greater than or equal to it.

Example 2:

Consider the project network depicted in Figure 1 but let $t_i \sim (1 - \alpha_i)\, F(x) + \alpha_i\, G(x)$ where $F(x)$, $G(x)$ are the distribution functions of $N(0,1)$ and $Exp\,(1)$ respectively and $\alpha_i = \frac{1}{i+1}$, $i = 1, 2, \ldots 7$.

Since the possible critical paths are $J_1 = \{0, 2, 1, 4\}$, $J_2 = \{0, 2, 3, 4\}$, $J_3 = \{0, 2, 4\}$, $J_4 = \{0, 1, 4\}$, we have

$$\lim_{\beta \to 1^-} \frac{U'_{2,k}(\beta)}{U'_{k,2}(\beta)} > 1 \qquad \text{for } k = 1, 3, 5$$

So J_2 is the desired critical path.

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